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Convergence of Gauss–Christoffel Formula with Preassigned Node for Cauchy Principal-Value Integrals*

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The authors consider Gauss–Christoffel formulas with preassigned node 0 for evaluating Cauchy singular integrals with an even generalized smooth Jacobi weight. A convergence theorem is given, and some asymptotic estimates of the remainder are established.

1. INTRODUCTION

Let $\Phi(f; t)$ be a Cauchy principal-value (V.P.) integral of the function f, namley,

$$\Phi(f; t) = \int_{-1}^{1} \frac{f(x)}{x-t} w(x)\, dx$$

$$= \lim_{\varepsilon \to 0+} \left\{ \int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right\} \frac{f(x)}{x-t} w(x)\, dx, \qquad |t| < 1, \tag{1.1}$$

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where

(i) $f \in TD := \{f \in C^0(I) / \int_0^1 u^{-1} \omega(f; u)\, du < \infty\}$. Here $I = [-1, 1]$ and $\omega(f; \ )$ is the modulus of continuity of f in I;

(ii) $w(x) := \psi(x)(1-x^2)^\mu$; $\mu > -1$, $\psi(x) = \psi(-x)$, $0 \leqslant \psi \in TD$, $1/\psi$ is (Lebesgue) integrable on I.

It is well known that (i) and (ii) imply the existence and continuity of $\Phi(f; t)$ moreover [2]:

$$\omega(\Phi f; \delta) = O(\omega(f; \delta)) \qquad (\delta \to 0).$$

In order to approximate Φf, we construct a Gauss–Christoffel type formula Φ_{2m} with a fixed node at 0 of a given multiplicity $2s$. This formula is a generalization of those contained in [11, 6].

The principal aim of this work is to examine the convergence of the sequence $\{\Phi_{2m} f\}$ under the assumption that the function $f \in TD$ has derivatives of whatever order will be needed at 0. We prove there is a subsequence $\{\Phi_{2m_n} f\}$ that converges uniformly to Φf on some closed subset of $(-1, 1) - \{0\}$. However, there exist also divergent subsequences $\{\Phi_{2m_v} f\}$. Consequently, one concludes that when the function f is not sufficiently smooth, this type of quadrature formula is unsuitable for applications.

We also prove that if we omit in $\phi_{2m} f$ a term corresponding to a node nearest to the singularity, then we obtain a quadrature formula $\Phi^*_{2m} f$ $(m \in N)$ that converges uniformly on a closed subset of $(-1, 1) - \{0\}$ to Φf, under the previous assumptions on $f \in TD$.

Moreover, we determine some asymptotic estimates of the remainders corresponding to the formulas Φ_{2m} and Φ^*_{2m}.

2. A Gauss–Christoffel Type Formula for the Evaluation of Principal Value Integral

Let

$$v(x) = x^{2s} w(x) \tag{2.1}$$

where $s \in N$ and w is defined by (ii).

Moreover, let $\{p_n\}$ be the sequence of the orthogonal polynomials in I associated with the weight fuction v defined by (2.1). Then, we denote the zeros of $p_n(x)$ by $x_{n,i} = x_{n,i}(v)$ $(i = 1, 2, ..., n)$, and the corresponding Christoffel numbers by $\lambda_{n,i} = \lambda_{n,i}(v)$ $(i = 1, 2, ..., n)$. The function v is a "generalized smooth Jacobi" weight $(v \in GSJ)$, and the properties of the orthogonal polynomials p_n have been extensively studied by Stancu [13], Rothmann [12], Badkov [1] and Nevai [9].

Now, we consider the Gauss–Christoffel quadrature formula with respect to the weight function w with the fixed knot 0, given with its order of multiplicity $2s$ [13]

$$G_{2m}(g)=\sum_{i=1}^{2m} A_{2m,i}\, g(x_{2m,i})+\sum_{j=0}^{s-1} B^{\sim}_{2m,2j}\, g^{(2j)}(0) \tag{2.2}$$

where $x_{2m,i}=x_{2m,i}(v)$ $(i=1, 2,..., 2m)$, and

$$A_{2m,i}=\frac{\lambda_{2m,i}(v)}{x^{2s}_{2m,i}(v)}, \qquad (i=1, 2,..., 2m), \tag{2.3}$$

$$B^{\sim}_{2m,2j}=\int_{-1}^{1} \beta_{2m,2j}(x)\, w(x)\, dx, \qquad (j=0, 1,..., s-1),$$

$$\beta_{2m,2j}(x)=\frac{x^{2j}}{(2j)!}\, p_{2m}(x) \sum_{k=0}^{s-j} \frac{x^{2k}}{(2k)!}\left[\frac{1}{p_{2m}(x)}\right]^{(2k)}_{x=0}, \tag{2.4}$$

and where the function g is defined on I. As the weight function v is even, we have

$$x_{2m,i}=-x_{2m,2m-i+1}, \qquad (i=1, 2,..., m),$$

$$A_{2m,i}=A_{2m,2m-i+1}, \qquad (i=1, 2,..., m).$$

Let $R_{2m}(g)$ be the remainder that is defined by

$$\int_{-1}^{1} g(x)\, w(x)\, dx=G_{2m}(g)+R_{2m}(g) \tag{2.5}$$

It is well known that the quadrature formula (2.2) has degree of exactness $4m+2s-1$.

Now, we may construct a Gauss–Christoffel quadrature formula for the approximate evaluation of Φf by the formula (2.2). In order to approximate the integral Φf, we write

$$\Phi(f;t)=f(t)\,⨍_{-1}^{1} \frac{w(x)}{x-t}\, dx+\int_{-1}^{1} \frac{f(x)-f(t)}{x-t}\, w(x)\, dx. \tag{2.6}$$

Hence, by (2.5) we have

$$\Phi(f;t)=f(t)\,⨍_{-1}^{1} \frac{w(x)}{x-t}\, dx+G_{2m}\left(\frac{f-f(t)}{e_1-t}\right)+R_{2m}\left(\frac{f-f(t)}{e_1-t}\right), \tag{2.7}$$

where $e_k(x)=x^k$ $(k\in N)$, and we have assumed that $t\neq 0$. Then (2.7) can be rewritten in the following form

$$\Phi(f;t)=\Phi_{2m}(f;t)+E_{2m}(f;t), \tag{2.8}$$

where we assume

$$E_{2m}(f; t) = R_{2m}\left(\frac{f-f(t)}{e_1 - t}\right), \tag{2.9}$$

$$\Phi_{2m}(f; t) = f(t) \int_{-1}^{1} \frac{w(x)}{x-t}\,dx + \sum_{i=1}^{2m} A_{2m,i} \frac{f(x_{2m,i}) - f(t)}{x_{2m,i} - t}$$
$$+ \frac{B_{2m,0}(t)}{t} f(t) - \sum_{j=0}^{2s-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0), \tag{2.10}$$

$$\left.\begin{aligned} B_{2m,2s-2}(t) &= B^{\sim}_{2m,2s-2} \\ B_{2m,j}(t) &= B^{\sim}_{2m,j} + (j+1)\frac{B_{2m,j+1}(t)}{t} \qquad (j = 2s-3, ..., 1, 0), \end{aligned}\right. \tag{2.11}$$

where $B^{\sim}_{2m,2j+1} = 0$. Moreover, by

$$\int_{-1}^{1} \frac{w(x)}{x-t}\,dx - \sum_{i=1}^{2m} \frac{A_{2m,i}}{x_{2m,i} - t} + \frac{B_{2m,0}(t)}{t} = R_{2m}\left(\frac{1}{e_1 - t}\right) := A_{2m}(t),$$

(2.10) becomes

$$\Phi_{2m}(f; t) = A_{2m}(t) f(t) + \sum_{i=1}^{2m} \frac{A_{2m,i}}{x_{2m,i} - t} f(x_{2m,i}) - \sum_{j=0}^{2s-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0). \tag{2.12}$$

We have established (2.12) under the assumption that $t \neq 0$; however, by an easy limit calculus, we may have the following quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{x} w(x)\,dx \cong \Phi_{2m}(f; 0) = \sum_{i=1}^{m} \frac{A_{2m,i}}{x_{2m,i}} [f(x_{2m,i}) - f(-x_{2m,i})]$$
$$+ \sum_{j=0}^{s-1} \frac{B_{2m,2j}}{2j+1} f^{(2j+1)}(0). \tag{2.13}$$

We note that formulas (2.12) and (2.13) have degree of exactness $4m + 2s$, but (2.12) depends on $m + 2s$ coefficients, whereas only $m + s$ coefficients are in (2.13); furthermore, the calculation of these is easier.

Finally, we observe that in the special case in which $w \equiv 1$ and $s = 1$, the formula (2.13) becomes the formula given by Hunter in [6], and for $w \equiv 1$ and $s = 0$, we obtain the Piessen's formula [11].

Now, we want examine the convergence of the formulas that we have introduced. It is clear that the convergence of the formula (2.13) follows easily; in fact, the function f has a derivative at point 0 and so the function $(f(x))/x$ is Riemann-integrable. Therefore, we shall attend to the convergence of the quadrature formula (2.10).

3. Notations and Basic Lemmas

The symbol "const" stands for some positive constant taking a different value each time it is used. If A and B are two expressions depending on some variables, then we write

$$A \sim B \qquad \text{if} \quad |AB^{-1}| \leqslant \text{const} \qquad \text{and} \qquad |A^{-1}B| \leqslant \text{const}$$

uniformly for the variables under consideration. Throughout this paper, $C_s(I)$ denotes the class of the continuous functions on I with $2s-2$ derivatives at point 0, and Λ denotes a closed set such that $\Lambda \subset (-1, 1) - \{0\}$.

As mentioned above, v is a generalized smooth Jacobi weight function; the properties of the corresponding orthogonal polynomials and of the Christoffel numbers are well known [1, 9].

From among these, we recall the important relations

$$\theta_{2m,i} - \theta_{2m,i+1} \sim (2m)^{-1}, \qquad (i = 1, 2, \ldots, 2m-1), \tag{3.1}$$

where $x_{2m,i} = \cos\theta_{2m,i}$ $(i = 1, 2, \ldots, 2m)$ are the zeros of p_{2m}, so ordered

$$-1 < x_{2m,1} < x_{2m,2} < \cdots < x_{2m,2m} < 1,$$

(see [9, Theorem 9.22, p. 166]), and

$$\lambda_{2m,i}(v) \sim (2m)^{-1}(1 - x_{2m,i}^2)^{\mu+1/2}(|x_{2m,i}| + (2m)^{-1})^{2s}, \qquad (i = 1, 2, \ldots, m). \tag{3.2}$$

(See [9, Theorem 6.3.28, p. 120].)

Then, we set: $N^* = \{m \in N / x_{2m,i} \neq t,\ i = 1, 2, \ldots, 2m\}$, and $x_{2m,c}$ denotes the closest knot to t; more precisely

$$|t - x_{2m,c}| = \min\{t - x_{2m,d},\ x_{2m,d+1} - t\},$$

where: $x_{2m,d} \leqslant t \leqslant x_{2m,d+1}$, $(0 \leqslant d \leqslant 2m)$. It is obvious that

$$N^{\sim} = \{m \in N^* / t - x_{2m,d} \sim x_{2m,d+1} - t\} \subseteq N^*,$$

and both sets are infinite [3].

At this point we prove the following lemmas, we need later for the main theorem.

LEMMA 3.I. *The inequalities*

$$\frac{\lambda_{2m,i}}{|x_{2m,i}-t|}<\frac{A_{2m,i}}{|x_{2m,i}-t|}<\frac{C}{(1-t^2)}\{(2m)^{-1}|x_{2m,i}-t|^{-1}+(1-x_{2m,i}^2)^{-1/2}\}$$
$$(i=1, 2,..., 2m), \qquad (3.3)$$

hold for some constant $C>0$ independent of m.

Proof. By (2.3) and (2.4), we have

$$A_{2m,i}=\lambda_{2m,i}x_{2m,i}^{2s}\sim(2m)^{-1}(1-x_{2m,i}^2)^{\mu+1/2}\left(1+\frac{1}{2m\,|x_{2m,i}|}\right)^{2s},$$
$$(i=1, 2,..., 2m).$$

Moreover, as the weight function $v\in GSJ$ is even, and by (3.1), we obtain

$$\frac{1}{2m\,|x_{2m,i}|}\leqslant\frac{1}{2m\,|x_{2m,m}|}\sim 1, \qquad (i=1, 2,..., 2m).$$

Thus

$$A_{2m,i}\leqslant\text{const}\,(2m)^{-1}(1-x_{2m,i}^2)^{-1/2}, \qquad (i=1, ..., 2m),$$

from which the second inequality in (3.3) easily follows, and also first inequality, from

$$A_{2m,i}\leqslant\lambda_{2m,i}, \qquad (i=1, 2,..., 2m). \quad \blacksquare$$

LEMMA 3.II. *Let $x_{2m,c}$ be the closest knot to t. If we set*

$$\sigma_m^*(t)=\sum_{\substack{i=1\\ i\neq c}}^{2m}\frac{A_{2m,i}}{|x_{2m,i}-t|} \qquad (3.4)$$

then

$$\sigma_m^*(t)\sim\log m, \qquad (m\in N), \qquad (3.5)$$

holds uniformly on A.

Proof. Without any loss of generality, we suppose that

$$x_{2m,c}=x_{2m,d}\leqslant t<x_{2m,d+1}; \qquad \text{thus} \quad t-x_{2m,d-1}\sim x_{2m,d+1}-t\sim(2m)^{-1}.$$

Then, by (3.3), we have

$$\sigma_m^*(t)>\sum_{\substack{i=1\\ i\neq d}}^{2m}\frac{\lambda_{2m,i}}{|x_{2m,i}-t|}=\sum_{i=1}^{d-1}\frac{\lambda_{2m,i}}{t-x_{2m,i}}+\sum_{i=d+1}^{2m}\frac{\lambda_{2m,i}}{x_{2m,i}-t}. \qquad (3.6)$$

Further, the function $u(x) = |x - t|^{-1}$ is such that

$$u^{(i)}(x) > 0, \qquad x \in [-1, x_{2m,d-1}], \qquad i = 0, 1, \ldots,$$

and

$$(-1)^i u^{(i)}(x) > 0, \qquad x \in [x_{2m,d+1}, 1], \qquad i = 0, 1, \ldots.$$

Then, by the generalized Markov–Stieltjes inequalities (see Lemma I.5.3 in [5, p. 30] and Lemmas 3.2, 3.3 in [8, p. 222]), we obtain

$$\sum_{i=1}^{d-1} \frac{\lambda_{2m,i}}{t - x_{2m,i}} \geqslant \int_{-1}^{x_{2m,d-1}} \frac{v(x)}{t - x}\, dx,$$

$$\sum_{i=d+1}^{2m} \frac{\lambda_{2m,i}}{x_{2m,i} - t} \geqslant \int_{x_{2m,d+1}}^{1} \frac{v(x)}{x - t}\, dx,$$

From these inequalities, we easily get

$$\sigma_m^*(t) > 2v(t) \log m + v(t) \log(1 - t^2) + V(t),$$

where

$$V(t) = \int_{-1}^{1} \left| \frac{v(x) - v(t)}{x - t} \right| dx.$$

From $v \in TD$, we have that the function V is continuous on A, and thus

$$\sigma_m^*(t) > a_1 \log m + b_1 \qquad (\forall t \in A),$$

where a_1, b_1 are two constants, independent of m.

Further, by (3.3) we obtain

$$\sigma_m^*(t) \leqslant \frac{\text{const}}{1 - t^2} \left[\sum_{\substack{i=1 \\ i \neq c}}^{2m} \frac{1}{2m\, |x_{2m,i} - t|} + \sum_{i=1}^{2m} \frac{1}{2m \sqrt{1 - x_{2m,i}^2}} \right].$$

By (3.1), we represent the two summations in the last inequality as Riemann–Darboux ones. Thus

$$\sigma_m^*(t) \leqslant \frac{\text{const}}{1 - t^2} \left[\int_{-1}^{x_{2m,d-1}} \frac{dx}{t - x} + \int_{x_{2m,d+1}}^{1} \frac{dx}{x - t} + \pi \right].$$

Again, by (3.1) and $t \in A$, we have

$$\sigma_m^*(t) \leqslant a_2 \log m + b_2,$$

where a_2, b_2 are two constants, independent of m.

Hence the lemma is proved. ∎

LEMMA 3.III. *For the coefficients* $B^{\sim}_{2m,2r}$ *in* (2.2) *the inequalities*

$$|B^{\sim}_{2m,2r}| \leqslant \frac{K}{(2r)!\, m^{2r+1}}, \qquad (r=0, 1, \ldots, s-1), \tag{3.7}$$

hold, where K *is a constant, independent of* m *and* r.

Proof. If one set $g(x)=x^{2r}$ $(r=0, 1, \ldots, s-1)$ in (2.5), we obtain

$$B^{\sim}_{2m,2r} = \frac{1}{(2r)!} \int_{-1}^{1} x^{2r} w(x)\, dx - \sum_{i=1}^{2m} \frac{\lambda_{2m,i}}{x_{2m,i}^{2(s-r)}}.$$

Now, the function $\gamma(x)=x^{2(r-s)} (r<s)$ is such that

$$\gamma^{(i)}(x)>0, \qquad x<0, \qquad i=0, 1, \ldots,$$

$$(-1)^{i}\gamma^{(i)}(x)>0, \qquad x>0, \qquad i=0, 1, \ldots,$$

and

$$v(x)\, \gamma(x) = x^{2r} w(x).$$

Then, by the generalized Markov–Stieltjes inequalities (see [8, p. 222]), we have

$$\sum_{i=1}^{m-1} \frac{\lambda_{2m,i}}{x_{2m,i}^{2(s-r)}} \leqslant \int_{-1}^{x_{2m,m}} x^{2r} w(x)\, dx \leqslant \sum_{i=1}^{m} \frac{\lambda_{2m,i}}{x_{2m,i}^{2(s-r)}},$$

$$\sum_{i=m+2}^{2m} \frac{\lambda_{2m,i}}{x_{2m,i}^{2(s-r)}} \leqslant \int_{x_{2m,m+1}}^{1} x^{2r} w(x)\, dx \leqslant \sum_{i=m+1}^{2m} \frac{\lambda_{2m,i}}{x_{2m,i}^{2(s-r)}}.$$

From these inequalities, we deduce

$$\alpha_m - \beta_m \leqslant B^{\sim}_{2m,2r} \leqslant \alpha_m,$$

where

$$\alpha_m = \frac{1}{(2r)!} \int_{x_{2m,m}}^{x_{2m,m+1}} x^{2r} w(x)\, dx,$$

and

$$\beta_m = \frac{2}{(2r)!} \frac{\lambda_{2m,m+1}}{x_{2m,m+1}^{2(s-r)}}.$$

Thus

$$|B^{\sim}_{2m,2r}| \leqslant \max\{\alpha_m, \beta_m\}.$$

Further, we have

$$(2r)!\,\alpha_m = w(\xi)\int_{x_{2m,m}}^{x_{2m,m+1}} x^{2r}\,dx = w(\xi)\,\frac{x_{2m,m+1}^{2r+1} - x_{2m,m}^{2r+1}}{2r+1}$$
$$= 2\omega(\xi)\,x_{2m,m+1}^{2r+1} \sim m^{-2r-1},$$

where $\xi \in (x_{2m,m}, x_{2m,m+1})$. Moreover, by (3.2), we see

$$(2r)!\,\beta_m \sim m^{-1}(1-x_{2m,m+1}^2)^{\mu+1/2}\left(x_{2m,m+1}+\frac{1}{2m}\right)^{2s} x_{2m,m+1}^{2(r-s)}$$
$$= m^{-1}(1-x_{2m,m+1}^2)^{\mu+1/2}\left(x_{2m,m+1}+\frac{1}{2m}\right)^{2r}$$
$$\times\left(x_{2m,m+1}+\frac{1}{2m}\right)^{2s-2r} x_{2m,m+1}^{2(r-s)}$$
$$= m^{-1}(1-x_{2m,m+1}^2)^{\mu+1/2}\left(x_{2m,m+1}+\frac{1}{2m}\right)^{2r}\left(1+\frac{1}{2mx_{2m,m+1}}\right)^{2s-2r}$$
$$\sim m^{-2r-1}$$

Hence, the lemma is proved. ▮

Before proceeding any further, we observe that the result of Lemma 3.III is sufficient to prove the convergence of the formula (2.2) when $g \in C^0(I)$ and we suppose the existence of $2s-1$ derivatives of the function g at point 0.

Now, we note that the functions $B_{2m,j}(t)$ depend on $B_{2m,j}^{\sim}$ by (2.11). From these relations, the equalities

$$B_{2m,2j}(t) = \frac{1}{(2j)!}\sum_{i=1}^{s-j}(2j+2i-2)!\,\frac{B_{2m,2j+2i-2}^{\sim}}{t^{2i-2}}, \qquad (j=0, 1,..., s-1),$$
$$B_{2m,2j+1}(t) = \frac{1}{t(2j+1)!}\sum_{i=1}^{s-j-1}(2j+2i)!\,\frac{B_{2m,2j+2i}^{\sim}}{t^{2i-2}}, \qquad (j=0, 1,..., s-2),$$

easily follow.

Thus, by (3.7) we deduce

$$\left|\frac{B_{2m,j}(t)}{t}\right| \leqslant \frac{K}{j!\,m^{j+1}}, \qquad (j=0, 1,..., 2s-2), \tag{3.8}$$

where $K>0$ is a constant, independent on m, j, and $t \in A$.

4. On the Convergence of Rule (2.10)

We set

$$P_{2s-2}(x)=\sum_{k=0}^{2s-2}\frac{f^{(k)}(0)}{k!}\,x^k,$$

$$g=f-P_{2s-2},$$

$$r_m=g-q_m,$$

where q_m is the best approximation polynomial with respect to the function g. We remark that $\omega(g;\delta)\leqslant \text{const}\ \omega(f;\delta)$ when $f\in C^0(I)$.

Under these assumptions, we may prove the following.

Lemma 4.I. *Given any function $f\in C_s(I)$, there is a constant L independent on f and $m\in N$ such that*

$$|E_{2m}(f;t)|\leqslant L\delta_m+|a_{2m,c}(t)|,\qquad t\in\Delta, \tag{4.1}$$

where

$$a_{2m,c}(t)=A_{2m,c}\,\frac{r_m(x_{2m,c})-r_m(t)}{x_{2m,c}-t},\qquad (m\in N^*),$$

$$\delta_m=\|r_m\|\,(\log m+m^{-1})+\int_0^{1/m}\frac{\omega(f;u)}{u}\,du+\frac{\|q'_m\|_\Delta}{m}$$

$$+\,m^{-1}\sum_{j=1}^{2s-2}\frac{|q_m^{(j)}(0)|}{j!\,m^j},\qquad (m\in N),$$

and Δ is a closed set such that $\Delta\subset(-1,1)$.

Proof. Having rule Φ_{2m} degree of exactness $4m+2s$, we may claim that

$$|E_{2m}(f;t)|\leqslant I_1+I_2+I_3,$$

where

$$I_1=|\Phi(r_m-r_m(t);t)|, \tag{4.2}$$

$$I_2=\left|\sum_{i=1}^{2m}A_{2m,i}\,\frac{r_m(x_{2m,i})-r_m(t)}{x_{2m,i}-t}\right|, \tag{4.3}$$

$$I_3=\left|\frac{B_{2m,0}(t)}{t}\,(r_m(t)-r_m(0))+\sum_{j=1}^{2s-2}\frac{B_{2m,j}(t)}{t}\,q_m^{(j)}(0)\right|. \tag{4.4}$$

Now, we obtain

$$I_1 \leqslant 2\,\|r_m\|\,W(t) + w(t)$$
$$\times \left\{4\,\|r_m\|\log m + \left|\int_{|x-t| \leqslant 1/m} \frac{r_m(x) - r_m(t)}{x-t}\,dx\right|\right\},$$

where

$$W(t) = \int_{-1}^{1} \left|\frac{w(x) - w(t)}{x-t}\right| dx$$

By

$$\left|\int_{|x-t| \leqslant 1/m} \frac{r_m(x) - r_m(t)}{x-t}\,dx\right| \leqslant \text{const} \int_0^{1/m} \omega(f; u)\,u^{-1}\,du + \frac{\|q'_m\|_\Delta}{m},$$

as $w \in TD$ implies $W \in C^0(\Delta)$, we obtain

$$I_1 \leqslant \text{const} \left\{\|r_m\|\,(\log m + 1) + \int_0^{1/m} \omega(f; u)\,u^{-1}\,du + \frac{\|q'_m\|_\Delta}{m}\right\}.$$

Furthermore, because of Lemma 3.II,

$$I_2 \leqslant \text{const}\,\|r_m\|\log m + |a_{2m,c}(t)|.$$

Finally, by (3.8), we have

$$I_3 \leqslant Km^{-1}\left\{2\,\|r_m\| + \sum_{j=1}^{2s-2} \frac{q_m^{(j)}(0)}{j!\,m^j}\right\}.$$

The combination of these inequalities proves the lemma. ∎

At this point, we may prove the following

THEOREM 4.I. *For any function $f \in C_s(I) \cap TD$, there exists a subsequence $\{\Phi_{2m_k}\}_{k \in N}$ uniformly convergent to Φf on Δ.*

Proof. First we remark that from $f \in C_s(I) \cap TD$ follows

$$\|r_m\| \leqslant \text{const}\,\omega(f; m^{-1}),$$

$$\lim_{m \in N} \int_0^{1/m} \omega(f; u)\,u^{-1}\,du = 0,$$

and

$$\lim_{m \in N} \omega(f; m^{-1}) \log m = 0.$$

Furthermore, it is well known [7] that for any function $F \in C^{(r)}(I)$, $(r \geqslant 0)$, if we denote by q_m the best approximation polynomial with respect to the function F, for any $k > r$ there exists a constant M_k such that

$$\|q_m^{(k)}\|_\Delta \leqslant M_k m^{k-r}\ \omega(F^{(r)}; m^{-1}), \tag{4.5}$$

where Δ is a closed set such that $\Delta \subset (-1, 1)$.
Thus, from $g \in C^0(I)$, we obtain

$$m^{-1}\left\{\|q'_m\|_\Delta + \sum_{j=1}^{2s-2} \frac{|q_m^{(j)}(0)|}{j!\, m^j}\right\} \leqslant \text{const}\ \omega(f; m^{-1}).$$

Finally, we have

$$\lim_{m \in N} \delta_m = 0.$$

At this point, we introduce the set $N' = \{n \in N^* / |x_{2n,c} - t| \sim n^{-1} \log^{-1} n\}$. By $N^{\sim} \subset N' \subset N^*$, we obtain that N' is an infinite set and $N' = \{m_k\}_{k \in N}$. Then, for any sufficiently large $k \in N$, there exists a constant $H > 0$, independent of f and k such that

$$\|a_{2m_k,c}\| \leqslant H\omega(f; m_k^{-1}) \log m_k = o(1), \qquad (k \to \infty),$$

and the theorem is proved. ∎

Now, let $LD(\lambda)$, $(\lambda > 0)$, be the class of functions $f \in C^0(I)$, such that $\omega(f; \delta) \log^{\lambda} \delta^{-1} = o(1)$, $(\delta \to 0^+)$. Obviously, we have

$$LD(\lambda) \supset TD \qquad \text{if} \quad \lambda \in (0, 1],$$

and

$$LD(\lambda) \subset TD \qquad \text{if} \quad \lambda > 1.$$

Note that by Lemma 4.I and Theorem 4.I the corollaries immediately follows:

COROLLARY 4.I. *For any function $f \in LD(\lambda)$, $(\lambda > 1)$, there exists a subsequence $\{E_{2m_i} f\}_{i \in N}$ such that*

$$\|E_{2m_i} f\| = o(\log^{\lambda - 1} m_i), \qquad (i \to \infty).$$

COROLLARY 4.II. *For any function $f \in Lip_M \alpha$, there exists a subsequence $\{E_{2m_j} f\}_{j \in N}$ such that*

$$\|E_{2m_j} f\| = O(m_j^{-\alpha} \log m_j), \qquad (j \to \infty).$$

COROLLARY 4.III. *For any function* $f \in Lip_M 1$, we have

$$\|E_{2n} f\| = O(n^{-1} \log n), \qquad (n \in N^*).$$

COROLLARY 4.IV. *For any function* $f \in C^{(k)}(I)$, *we have*

$$\|E_{2n} f\| = O(n^{-k} \log n\omega(f; n^{-1})), \qquad (n \in N^*).$$

Furthermore, we observe that obvious changes in the proof of Theorem 4.I are sufficient to prove the following:

THEOREM 4.II. *If the integral* Φf *exists, then, for any function* $f \in C_s(I) \cap LD(1)$, *there exists a subsequence* $\{\Phi_{m_k} f\}_{k \in N}$ *convergent to* Φf *in* $(-1, 1) - \{0\}$.

At this point, we remark that by Lemma 3.II the norm of Φ_m is not bounded; then the continuity of the function f is not sufficient for the convergence of the sequence $\{\Phi_{2m}\}$, which besides is defined just for $m \in N^* \subset N$. Yet, in some cases we have $N^{\sim} = N^*$. This is true when $s = 0$, $\psi(x) \equiv 1$, $\mu = \pm\frac{1}{2}$, and $t = \cos(\pi p/q)$, where p/q is a rational number in $(0, 1)$, [8].

Further, it is not difficult to find a subsequence $\{\Phi_{2m_\nu}\}_{\nu \in N}$ not convergent when the function $f \in Lip_M \alpha$, $(\alpha < 1)$. In fact, following the example $s = 0$, $\psi(x) \equiv 1$, $\mu = \pm\frac{1}{2}$, if we suppose $t = \cos(\theta\pi)$, where θ is an irrational number in $(0, 1)$, we have $|a_{2m_\nu, c}| \sim (2m_\nu)^{1-\alpha}$ for a particular sequence $\{m_\nu\}_{\nu \in N} \subset N^*$ (see [10, p. 23]).

From the proof of Theorem 4.I, we have that the term that causes difficulties to the convergence of $\Phi_{2m} f$ is $a_{2m,c}(t)$, corresponding to the closest knot to the singularity. Now, let us omit this term and consider the quadrature formula

$$\Phi^*_{2m}(f; t) = f(t) \int_{-1}^{1} \frac{w(x)}{x - t}\, dx + \sum_{i=1}^{2m} A_{2m,i} \frac{f(x_{2m,i}) - f(t)}{x_{2m,i} - t} + \frac{B_{2m,0}(t)}{t} f(t) - \sum_{j=0}^{2s-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0), \tag{4.6}$$

that can be rewritten in the following form

$$\Phi^*_{2m}(f; t) = A^*_{2m}(t) f(t) + \sum_{\substack{i=1 \\ i \neq c}}^{2m} \frac{A_{2m,i}}{x_{2m,i} - t} f(x_{2m,i}) - \sum_{j=0}^{2s-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0), \tag{4.7}$$

where

$$A^*_{2m}(t) = \int_{-1}^{1} \frac{w(x)}{x - t}\, dx - \sum_{\substack{i=1 \\ i \neq c}}^{2m} \frac{A_{2m,i}}{x_{2m,i} - t} + \frac{B_0(t)}{t}.$$

Then, the corresponding remainder is defined by

$$E^*_{2m}(f;t)=\Phi(f;t)-\Phi^*_{2m}(f;t).$$

One can easily prove that Φ^*_{2m} has degree of exactness 0, however we may consider Φ^*_{2m} for any $m\in N$, and we may prove the following:

THEOREM 4.III. *For any function $f\in C_s(I)\cap TD$, the sequence $\{\Phi^*_{2m}f\}_{m\in N}$ converges uniformly to Φf on Λ.*

Proof. Proceeding as in [4], we obtain

$$E^*_{2m}(f;t)=E^*_{2m}(r_m;t)+A_{2m,c}\frac{q_m(x_{2m,c})-q_m(t)}{x_{2m,c}-t},$$

and recalling (3.4), (4.2), (4.4), we have

$$|E^*_{2m}(f;t)|\leqslant I_1+2\,\|r_m\|\,\sigma_m(t)+I_3+A^*_{2m,c}\,\|q'_m\|_\Delta.$$

At this point, if we proceed as for the proof of Theorem 4.I, we may obtain

$$\|E^*_{2m}f\|_A\leqslant C(\delta_m+\omega(f;m^{-1})), \tag{4.8}$$

where C is a constant, independent on f and m.

This completes the proof of the theorem. ∎

Furthermore, by (4.8) we obtain

$$\|E^*_{2m}f\|_A=o(\log^{\lambda-1}m) \qquad \text{if}\quad f\in LD(\lambda), \qquad (\lambda>1). \tag{4.9}$$

$$\|E^*_{2m}f\|_A=O(m^{-\alpha}\log m) \qquad \text{if}\quad f\in Lip_M\alpha, \qquad (0<\alpha\leqslant 1). \tag{4.10}$$

On the other side, we have the following:

THEOREM 4.IV. *If the integral Φf exists, then for any function $f\in C_s(I)\subset LD(1)$, the sequence $\{\Phi^*_{2m}f\}$ converges to Φf in $(-1,1)-\{0\}$:*

Further, we point out that Theorem 4.III, Theorem 4.IV and the relations (4.9), (4.10) also hold for the quadrature formula Φ^{**}_{2m} that we may obtain from Φ_{2m} omitting the two terms that correspond to both knots $x_{2m,d}$ and $x_{2m,d+1}$.

Finally, note that in the special case in which $s=0$, we obtain results established in [4].

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